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# A simple model for turbulent boundary layer mass transfer on flat plate in parallel flow

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**Abstract**

A simple model for turbulent mass transfer from a flat plate valid for wider range of Schmidt numbers is proposed. Based on $1/n$ power law velocity/concentration profiles and integral momentum/mass equations, the model reveals that $\lambda_c$, a parameter in universal concentration profile, is a function of $Sc$. Through an empirical approach, it is given as $\lambda_c = 8.55\,Sc^{0.37}$. With this function, the model matches with experimental data up to $Sc = 108$. 

*Keywords:* Turbulent mass transfer; Schmidt number; Momentum/mass equations

## 1. Introduction

The objective of this communication is to present a simplified semi-empirical approach to account for the mass transfer coefficient for the case of turbulent boundary layer formed over a soluble flat plate held parallel to the flow valid for a wide range of $Sc$. In laminar range, experimental data have been found to be in agreement with theoretical predictions based on integral boundary layer equation [1]. For turbulent flows, the analysis and the predictions are yet not conclusive because of the absence of a universal turbulent model. Correlations were proposed based on Chilton–Colburn analogy [2] and later corrected by Rubesin's starting length correction factor [3]. Several workers correlated turbulent heat transfer data using velocity and temperature profiles based on $1/n$ power law. Reynolds [4] showed that at $n = 7$, heat transfer data had better agreement. Interest in $1/n$ power law has been revived with extended range up to Reynolds numbers of 10 000 000 [5–7]. Mass transfer coefficient can be obtained through $1/n$ power law and the integral equation with the assumption of identical hydrodynamic and concentration boundary layer thickness, limiting the solution to $Sc = 1$ [8]. Recently Nassif et al. [9] used $1/n$ power law to correlate average mass transfer data for a naphthalene plate subliming in parallel flow of air. With the introduction of exponent on Schmidt number and relating the universal velocity constants to Reynolds number, they could extend the validity of their correlation up to $Sc = 2.5$.

In the present work, a more general correlation valid for a wider range of Schmidt numbers is proposed based on the mass transfer integral equation. The constant $\lambda_c$, in $1/n$ concentration profile, is a function of $\lambda$ and $Sc$ where $\lambda$ as defined in Eq. (1), is taken, as discussed elsewhere, to be a constant. A relation between $\lambda_c$ and $Sc$ has been obtained by correlating available heat and mass transfer data with the model.

## 2. Analysis and discussion

Consider a horizontal flat plate oriented in the direction of an incompressible, Newtonian and steady-state flow as shown in Fig. 1. Convective solid–fluid mass transfer is taking place at the plate surface after an initial inert length $x_0$. The universal velocity profile in the hydrodynamic boundary layer can be written as

$$u^+ = \lambda[y^+]^{1/n} \tag{1}$$

where

$$u^+ = \frac{u_x}{u^*}, \qquad u^* = \sqrt{\frac{\tau_w}{\rho}}, \qquad y^+ = \frac{u^* y}{\nu} \tag{2}$$

At $y = \delta$, the velocity equals the free stream velocity $U$. With this boundary condition $\lambda$ is eliminated from Eq. (1) to give

$$u_x = U\left(\frac{y}{\delta}\right)^{1/n} \tag{3}$$

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**Nomenclature**

| | |
|---|---|
| $B$, $D$ and $E$ | terms defined in Eqs. (14), (17) and (18) |
| $C_{\mathrm{A}}$ | concentration of component A |
| $C_{\mathrm{A}\infty}$ | bulk concentration of the free stream |
| $C_{\mathrm{A0}}$ | surface concentration of the plate |
| $C'_{\mathrm{A}}$ | $(C_{\mathrm{A}} - C_{\mathrm{A0}})$ |
| $C'^{+}_{\mathrm{A}}$ | dimensionless concentration given by Eq. (11) |
| $C_p$ | heat capacity |
| $D_{\mathrm{AB}}$ | binary diffusion coefficient |
| $h_x$ | local heat transfer coefficient |
| $k$ | thermal conductivity |
| $k_x$, $k_L$ | local and average mass transfer coefficients |
| $L$ | length of the plate |
| $n$ | exponent in the power law |
| $N_{\mathrm{A0}}$ | mass transfer flux |
| $Nu_x$ | local Nusselt number, $h_x x/k$ |
| $Pr$ | Prandtl number, $C_p\mu/k$ |
| $Re_x$, $Re_L$ | Reynolds number based on $x$ and $L$, $Ux/\nu$ and $UL/\nu$ |
| $Sc$ | Schmidt number, $\nu/D_{\mathrm{AB}}$ |
| $Sc_{\mathrm{t}}$ | turbulent Schmidt number |
| $Sh_L$ | average Sherwood number, $k_L L/D_{\mathrm{AB}}$ |
| $St_x$, $St_L$ | local and average Stanton numbers, $k_x/U$, $k_L/U$ |
| $u_x$ | local velocity in the boundary layer |
| $u^+$ | dimensionless velocity given by Eq. (2) |
| $U$ | free stream velocity |
| $x$ | axial distance |
| $x_0$ | starting inert length |
| $y$ | axial distance |
| $y^+$ | dimensionless distance for hydrodynamic analysis, $yu^*/\nu$ |
| $y^+_{\mathrm{c}}$ | dimensionless distance for mass transfer analysis, $yu^*/D_{\mathrm{AB}}$ |

*Greek letters*

| | |
|---|---|
| $\alpha$ | parameter defined in Eq. (24) |
| $\delta$ | hydrodynamic boundary layer thickness |
| $\delta_{\mathrm{c}}$ | concentration boundary layer thickness |
| $\varepsilon_{\mathrm{c}}$ | eddy concentration diffusivity |
| $\varepsilon_{\mathrm{m}}$ | eddy momentum diffusivity |
| $\lambda$ | constant in universal velocity profile |
| $\lambda_{\mathrm{c}}$ | constant in universal concentration profile |
| $\mu$ | fluid viscosity |
| $\nu$ | kinematic viscosity, $\mu/\rho$ |
| $\rho$ | fluid density |
| $\tau_{\mathrm{w}}$ | shear stress at wall |

$$u^* = U\lambda^{(-n/(n+1))}\left[\frac{\nu}{\delta U}\right]^{(1/(n+1))} \tag{4}$$

$$\tau_{\mathrm{w}} = \rho U^2\lambda^{(-2n/(n+1))}\left[\frac{\nu}{\delta U}\right]^{(2/(n+1))} \tag{5}$$

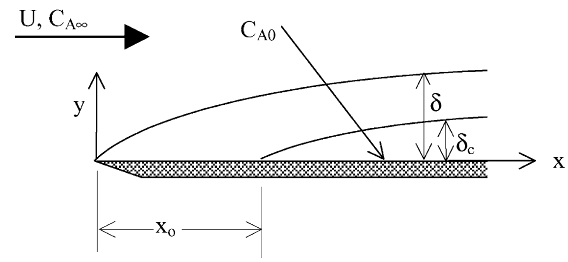


Fig. 1. Schematic diagram of momentum and concentration boundary layers.

The boundary layer thickness $\delta$ can be calculated by solving the momentum integral equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta}(U - u_x)u_x\,\mathrm{d}y = \frac{\tau_{\mathrm{w}}}{\rho} \tag{6}$$

together with Eq. (3) and the boundary condition at $x = 0$, $\delta = 0$. Thus, $\delta$ can be expressed as

$$\delta = \lambda^{(-2n/(n+3))}\left[\frac{(n+3)(n+2)x}{n}\right]^{((n+1)/(n+3))} \times\left[\frac{\nu}{U}\right]^{(2/(n+3))} \tag{7}$$

It should be noted that for the widely used values for $n = 7$ and $\lambda = 8.74$, Eq. (7) yields,

$$\delta = 0.371x\,Re_x^{-0.2} \tag{8}$$

which is in agreement with

$$\delta = 0.376x\,Re_x^{-0.2} \tag{9}$$

Eq. (9) was obtained from experimental data of Schultz-Grunow [10] (as quoted in Skelland [8]). Similar to velocity, a concentration profile is assumed

$$C'^{+}_{\mathrm{A}} = \lambda_{\mathrm{c}}[y^+_{\mathrm{c}}]^{1/n} \tag{10}$$

where the dimensionless concentration $C'^{+}_{\mathrm{A}}$ is defined as

$$C'^{+}_{\mathrm{A}} = \frac{C'_{\mathrm{A}}}{C'^{*}_{\mathrm{A}}} = \frac{(C_{\mathrm{A}} - C_{\mathrm{A0}})u^*}{N_{\mathrm{A0}}} \quad \text{using } C'^{*}_{\mathrm{A}} = \frac{N_{\mathrm{A0}}}{u^*} \tag{11}$$

A corresponding dimensionless distance $y^+_{\mathrm{c}}$ is defined as $y^+_{\mathrm{c}} = (yu^*/D_{\mathrm{AB}})$. At the edge of the boundary layer (i.e., $y = \delta_{\mathrm{c}}$), concentration is $C_{\mathrm{A}\infty}$. This condition with Eq. (10) gives $1/n$ power law profile for concentration.

$$\frac{C'_{\mathrm{A}}}{C'_{\mathrm{A}\infty}} = \left(\frac{y}{\delta_{\mathrm{c}}}\right)^{1/n} \tag{12}$$

Local Stanton number, $St_{\mathrm{AB}_x}$ can be obtained through Eqs. (4), (7), (10)–(12):

$$\frac{N_{\mathrm{A0}}}{UC'_{\mathrm{A}\infty}} = \frac{k_x}{U} = St_{\mathrm{AB}_x} = Bx^{((1-n)/n(n+3))}\delta_{\mathrm{c}}^{-1/n} \tag{13}$$

where

$$B = \lambda^{((1-n)/(n+3))}\lambda_c^{-1}\left[\frac{(n+3)(n+2)}{n}\right]^{((1-n)/n(n+3))} \times\left[\frac{\nu}{U}\right]^{((n-1)/n(n+3))}\left[\frac{D_{AB}}{U}\right]^{1/n} \quad (14)$$

The integral equation for mass transfer is

$$\frac{d}{dx}\int_0^{\delta_c}(C'_{A\infty}-C'_A)u_x\,dy = N_{A0} \quad (15)$$

Eqs. (3), (12) and (15) with boundary condition, at $x = x_0$, $\delta_c = 0$, yield

$$\delta_c = x^{((n+1)/(n+3))}\left[\frac{E(3+n)}{1-3D+n-nD}\right]^{(n/(n+2))} \times\left[1-\left(\frac{x_0}{x}\right)^{((2+n)(1-3D+n-nD)/n(n+3))}\right]^{(n/(n+2))} \quad (16)$$

Here

$$D = \frac{n}{3n+n^2} \quad (17)$$

and

$$E = (n+2)\lambda^{(-(n+1)/(n+3))}\lambda_c^{-1}\left(\frac{\nu}{U}\right)^{((n+1)/n(n+3))} \times\left(\frac{D_{AB}}{U}\right)^{1/n}\left(\frac{(n+2)(n+3)}{n}\right)^{(2/n(n+3))} \quad (18)$$

Final expression for $St_{AB_x}$ can be obtained by substituting $\delta_c$ from Eq. (16) into Eq. (13).

$$St_{AB_x} = \lambda^{((3-n^2)/(n+2)(n+3))}\lambda_c^{(-(n+1)/(n+2))}\,Sc^{(-(n+1)/n(n+2))} \times Re_x^{(-2/(n+3))}\left[\frac{(n+3)(n+2)}{1-3D+n-nD}\right]^{(-1/(n+2))} \times\left[\frac{(n+3)(n+2)}{n}\right]^{(-(n+1)/(n+2)(n+3))} \times\left[1-\left(\frac{x_0}{x}\right)^{((2+n)(1-3D+n-nD)/n(n+3))}\right]^{(-1/(n+2))} \quad (19)$$

In the present analysis, the values of $n$ and $\lambda$ were taken as 7 and 8.74, respectively, which are the most widely used values for fully developed turbulent flows [11]. Therefore, Eq. (19) becomes

$$St_{AB_x} = 0.1982\lambda_c^{-0.889}\,Sc^{-0.12698}\,Re_x^{-0.2}\left[1-\left(\frac{x_0}{x}\right)^{0.9}\right]^{-1/9} \quad (20)$$

The average $Sh$ is obtained by following integration using IMSL subroutine QDAGS

$$Sh_L = \frac{Sc\,Re_L}{(L-x_0)}\int_{x_0}^{L} St_{AB_x}\,dx. \quad (21)$$

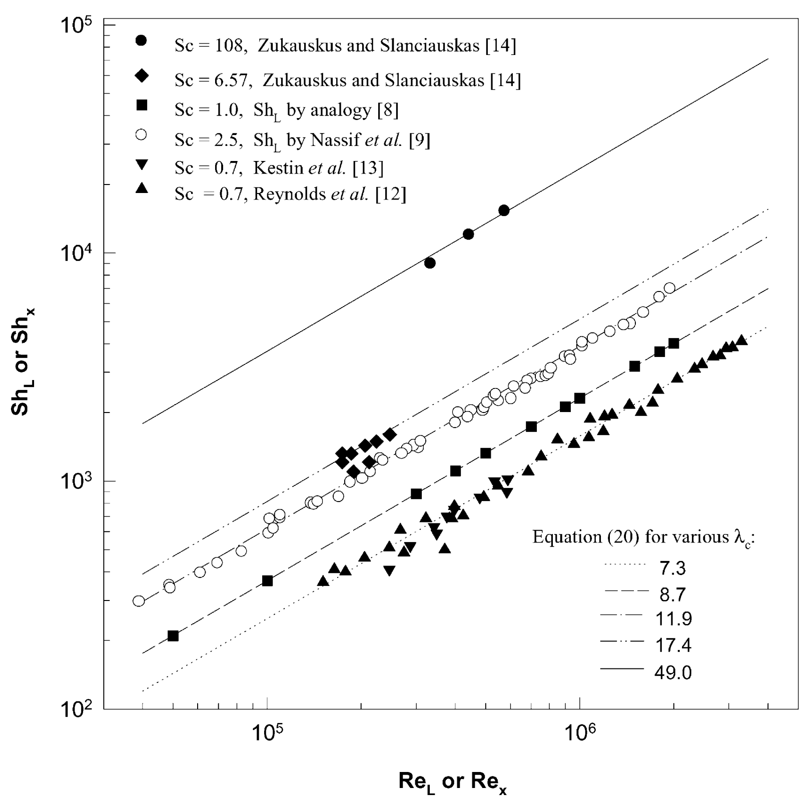


Fig. 2. Comparison of experimental data with Eq. (20). See Reynolds et al. [12], Kestin et al. [13], Zukauskus and Slanciauskas [14].

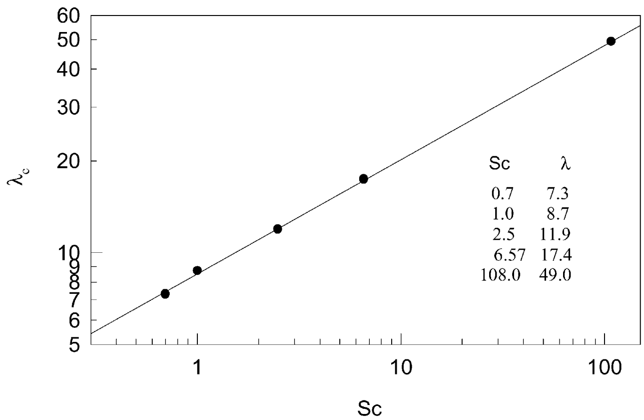


Fig. 3. Plot of parameter $\lambda_c$ versus *Sc*.

At this point, an explicit expression for $\lambda_c$ is required, which if obtained would render the whole analysis simple. Relevant data on flat plate (including those for heat transfer) at different values of *Sc* (or *Pr*) were matched with the present analysis using Eq. (20) or its averaged form. The matched plots are shown in Fig. 2. It should be noted that value of $\lambda_c$ was adjusted till a good agreement with the experimental data was established. Thus, different adjusted values of $\lambda_c$ were obtained for corresponding *Sc*. The desired expression for $\lambda_c$ is obtained by linear regression of adjusted $\lambda_c$ and *Sc*. The data as shown in Fig. 3, displays linearity and, therefore, can be easily extrapolated. The resulting equation is

$$\lambda_c = 8.55Sc^{0.37} \tag{22}$$

Dividing Eq. (22) by $\lambda = 8.74$ yields,

$$\frac{\lambda_c}{\lambda} = 0.98Sc^{0.37} \tag{23}$$

Eq. (23) is the equivalent of Eq. (22), but is more meaningful in its present format. If the constant in Eq. (23) is approximated to unity, the ratio $\lambda_c/\lambda$ becomes purely a function of *Sc*, a ratio of momentum and mass diffusivities. In principle, this ratio should be a function of turbulent Schmidt number ($Sc_t = \varepsilon_m/\varepsilon_c$). It should be noted that Eq. (23) qualitatively matches the equation quoted by Nassif et al. [9] who have postulated that

$$Sc^{\alpha} = \frac{\nu + \varepsilon_m}{D_{AB} + \varepsilon_c} \tag{24}$$

where $\alpha$ in the above equation covers the contribution due to eddy diffusivities. The authors obtained the values of $\alpha$ and $\lambda$ for $n = 7$ by comparing their analysis with an empirical heat transfer equation [15] valid for $0.5 < Pr < 1.0$. Later, $\alpha$ and $\lambda$ were generalized, through a very empirical approach, to be functions of *n* which itself is taken as function of $Re_L$. Therefore, the $\alpha$ and $\lambda$ depend on hydrodynamics only which is at variance with the assumption (implicit in Eq. (23)) that $\alpha$ depends also on $\varepsilon_c$. However, their analysis holds only for *Sc* between 1 and 2.5. At any other *Sc* widely off the limits of unity, a mismatch is expected. The predictions from the present model are compared with the models of Nassif et al. [9] and Churchill [16] in Fig. 4. The model of Nassif et al. expectedly overpredicts at higher values of *Sc*, while the present model is in coherence with Churchill's model. In addition, it is proposed that the present model would cover the predictions for a much wider range of *Sc* number. This is based on intuitive arguments that the transfer mechanism remains invariant even at high *Sc*. This implies that linear dependence of $\lambda$ with $Sc^{0.37}$ is sustained to give an extended validity of the present model.

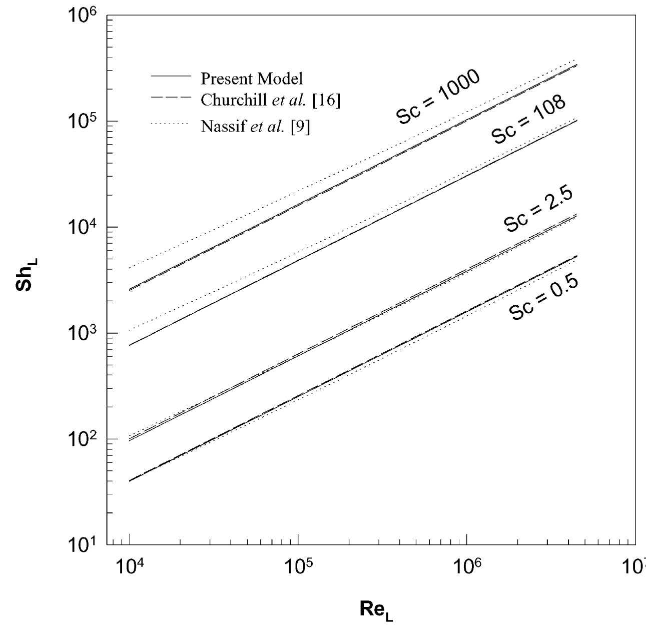


Fig. 4. Comparison of models at various values of *Sc*.

## 3. Conclusion

A simple semi-empirical model to predict mass transfer coefficient for fully developed boundary layer turbulent flow over a flat surface is proposed. For the widely used values of $n = 7$ and $\lambda = 8.74$, the model incorporates an empirical expression for $\lambda_c$, which together with integral equations and $1/n$ power law velocity and concentration profiles, calculates Sherwood numbers. Expression for $\lambda_c$ was obtained by matching experimental data for a range of *Sc* from 0.7 to 108. The validity of this function can be extended by comparing with data at higher *Sc*. Experiments are in progress using electrochemical limiting diffusion current technique which will provide data for *Sc* of the order of 2000.

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